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On the Ergodic Properties of Glauber Dynamics

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We show that if there is an infinite volume Gibbs measure which satisfies a logarithmic Sobolev inequality with local coefficients of moderate growth, then the corresponding stochastic dynamics decays to equilibrium exponentially fast in the uniform norm.

KEY WORDS: Glauber dynamics; logarithmic Sobolev inequality; ergodic properties.

INTRODUCTION

Throughout this article, we will be studying models of *finite-range, shift-invariant lattice gases*. To be more precise, the setting will be the following.

The Lattice. The lattice underlying our models will be the d-dimensional square lattice $\mathbb{Z}^d$ for some fixed $d \in \mathbb{Z}^+$; and, for $\mathbf{k} = (k^1,..., k^d) \in \mathbb{Z}^d$, we will use the norm $|\mathbf{k}| \equiv \max_{1 \leqslant i \leqslant d} |k^i|$. Given $\Lambda \subseteq \mathbb{Z}^d$, we will use $\Lambda \mathrm{C} \equiv \mathbb{Z}^d \backslash \Lambda$ to denote the complement of Λ, $|\Lambda|$ to denote the cardinality of Λ, and $\mathbf{k} + \Lambda$ to denote the translate $\{\mathbf{k} + \mathbf{j} : \mathbf{j} \in \Lambda\}$ of Λ by $\mathbf{k} \in \mathbb{Z}^d$. Finally, we will occasionally use the notation $\Lambda \subset\subset \mathbb{Z}^d$ to mean that $|\Lambda| < \infty$, and $\mathfrak{F}$ will stand for the set of all nonempty $\Lambda \subset\subset \mathbb{Z}^d$.

The Spin and Configuration Spaces. The spin space M for our model will be a finite set (with the discrete topology), and our configuration space will be the product space $\mathbf{M} \equiv M^{\mathbb{Z}^d}$ (endowed with the product topology). Given a nonempty $X \subseteq \mathbb{Z}^d$, we will use $\mathbf{x} \in \mathbf{M} \mapsto \mathbf{x}_X \in M^X$ to denote the natural projection taking $\mathbf{M}$ onto M^X, $B_X(\mathbf{M})$ and $C_X(\mathbf{M})$ to denote the sets of functions on $\mathbf{M}$ of the form $\mathbf{x} \in \mathbf{M} \mapsto \varphi(\mathbf{x}_X) \in \mathbb{R}$ as φ runs

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over, respectively, the set $B(M^X)$ of bounded, Borel-measurable and the set $C(M^X)$ of continuous functions on M^X; and $\mathscr{F}_X$ will denote the σ-algebra over $\mathbf{M}$ generated by the elements of $B_X(\mathbf{M})$. When $X=\{\mathbf{k}\}$, we will use $\mathbf{x}_\mathbf{k}$ in place of $\mathbf{x}_{\{\mathbf{k}\}}$; and when $X=\mathbb{Z}^d$, it is clear that $\mathscr{F}_X$ is precisely the Borel field $\mathscr{B}_\mathbf{M}$ over $\mathbf{M}$, and we will simply write $\mathscr{F}$ instead of $\mathscr{F}_{\mathbb{Z}^d}$. Also, we will say that a function $f\colon \mathbf{M}\to\mathbb{R}$ is *local* and will write $f\in C_0(\mathbf{M})$ if it is an element of $C_X(\mathbf{M})$ for some $X\in\mathfrak{F}$; and, for any bounded $f\colon \mathbf{M}\to\mathbb{R}$, $\|f\|_\mathrm{u}$ will be used to denote the *uniform norm* (i.e., "sup") norm of f. Finally, for each $\mathbf{k}\in\mathbb{Z}^d$, we define the *shift transformation* $\theta^\mathbf{k}\colon \mathbf{M}\to\mathbf{M}$ so that $(\theta^\mathbf{k}\mathbf{x})_\mathbf{j}=\mathbf{x}_{\mathbf{k}+\mathbf{j}}$ for every $\mathbf{x}\in\mathbf{M}$ and every $\mathbf{j}\in\mathbb{Z}^d$.

For various constructions, it will be convenient to introduce some additional notation. In the first place, given $\varnothing\neq X\subset\mathbb{Z}^d$, we define

$$(\mathbf{x}^X,\mathbf{y}^{X\mathrm{C}})\in M^X\times M^{X\mathrm{C}}\mapsto x^X\bullet y^{X\mathrm{C}}\in\mathbf{M}$$

so that $\mathbf{x}^X\bullet\mathbf{y}^{X\mathrm{C}}$ is the element $\mathbf{z}\in\mathbf{M}$ determined by

$$\mathbf{z}_X=\mathbf{x}^X \qquad\text{and}\qquad \mathbf{z}_{X\mathrm{C}}=\mathbf{y}^{X\mathrm{C}}$$

and, for $f\colon\mathbf{M}\to\mathbb{R}$ and $\mathbf{y}^{X\mathrm{C}}\in M^{X\mathrm{C}}$, we define $f(\cdot\mid\mathbf{y}^{X\mathrm{C}})$ on M^X and $f_X(\cdot\mid\mathbf{y}^{X\mathrm{C}})$ on $\mathbf{M}$ by

$$\mathbf{x}^X\in M^X\mapsto f(\mathbf{x}^X\mid\mathbf{y}^{X\mathrm{C}})\equiv f(\mathbf{x}^X\bullet\mathbf{y}^{X\mathrm{C}})$$

and

$$\mathbf{x}\in\mathbf{M}\mapsto f_X(\mathbf{x}\mid\mathbf{y}^{X\mathrm{C}})\equiv f(\mathbf{x}_X\bullet\mathbf{y}^{X\mathrm{C}})$$

Second, for $\mathbf{y}\in\mathbf{M}$, we write $f_X(\mathbf{x}\mid\mathbf{y})$ insteady of $f_X(\mathbf{x}\mid\mathbf{y}_{X\mathrm{C}})$; and, when $X=\{\mathbf{k}\}$ we will use $f_\mathbf{k}(\cdot\mid\mathbf{y})$ in place of $f_{\{\mathbf{k}\}}(\cdot\mid\mathbf{y})$. Since both

$$(\mathbf{x}^X,\mathbf{y}^{X\mathrm{C}})\in M^X\times M^{X\mathrm{C}}\mapsto\mathbf{x}^X\bullet\mathbf{y}^{X\mathrm{C}}\in\mathbf{M}$$

$$(\mathbf{x},\mathbf{y})\in\mathbf{M}^2\mapsto\mathbf{x}_X\bullet\mathbf{y}_{X\mathrm{C}}\in\mathbf{M}$$

are continuous maps, all the preceding constructions preserve both continuity and measurability.

Measures and Partial Differences. For nonempty $X\subseteq\mathbb{Z}^d$, we use $\mathfrak{M}_1(M^X)$ to denote the space of Borel probability measures μ on $(M^X,\mathfrak{B}_M^X)$, and give $\mathfrak{M}_1(M^X)$ the topology of weak convergence, namely $\{\mu_n\}_1^\infty\subseteq\mathfrak{M}_1(M^X)$ converges to μ, written $\mu_n\Rightarrow\mu$, means that

$$\int_{M^X}\varphi\,d\mu_n\to\int_{M^X}\varphi\,d\mu \qquad\text{for every}\quad \varphi\in C(M^X)$$

Next, given $\varphi \in B(M^X)$ and $\mu \in \mathfrak{M}_1(M^X)$, we will often write $\langle \varphi, \mu \rangle$ to denote $\int_{M^X} \varphi \, d\mu$; and, when $\varnothing \neq X \subset \mathbb{Z}^d$, $\mu \in \mathfrak{M}_1(M^X)$ and $\varphi \in B(\mathbf{M})$ $[\varphi \in C(\mathbf{M})]$, we define the $\mathscr{F}_{X^C}$-measurable (continuous) mapping

$$\mathbf{x} \in \mathbf{M} \mapsto \langle \varphi \rangle_{X,\mu}(\mathbf{x}) = \int_{M^X} f_X(\xi^X \mid \mathbf{x}) \, \mu(d\xi^X)$$

Throughout, we will reserve λ to denote the normalized counting measure on M. Further, will we write $\langle \varphi \rangle_X$ in place of $\langle \varphi \rangle_{X,\lambda^X}$ and $\langle \varphi \rangle_{\mathbf{k}}$ when $X = \{\mathbf{k}\}$. Finally, for each $Y \in \mathfrak{F}$, we define the *partial difference operator* $\partial_Y: B(\mathbf{M}) \to B(\mathbf{M})$ by

$$\partial_Y \varphi = \langle \varphi \rangle_Y - \varphi \tag{0.1}$$

and use $\partial_{\mathbf{k}}$ when $Y = \{\mathbf{k}\}$.

Potentials and Gibbs States. A *potential* is a family $\mathscr{J} = \{J_X : X \in \mathfrak{F}\}$ where, for each $X \in \mathfrak{F}$, $J_X \in C_X(\mathbf{M})$. Throughout, we will be assuming that our potentials are *shift invariant* in the sense that $J_{\mathbf{k}+X} = J_X \circ \theta^{\mathbf{k}}$, for all $X \in \mathfrak{F}$ and $\mathbf{k} \in \mathbb{Z}^d$. Further, we assume that $\mathscr{J}$ has *finite range* $R \in \mathbb{Z}^+$: that is, for each $X \in \mathfrak{F}$, $J_X \equiv 0$ if $\mathbf{0} \in X \not\subseteq [-R, R]^d$. Given $\mathscr{J}$, we define the corresponding *local specification* $\mathfrak{E} = \mathfrak{E}(\mathscr{J})$ to be the collection of Markov operators

$$[\mathbb{E}^X \varphi](\xi) = \int_M \varphi(\mathbf{y}) \, E^X(d\mathbf{y} \mid \xi)$$
$$= \frac{1}{Z_X(\xi_{X^C})} \int_M \varphi(\mathbf{y}_X \bullet \xi_{X^C}) \exp[-U^X(\mathbf{y}_X \bullet \xi_{X^C})] \, \boldsymbol{\lambda}(d\mathbf{y})$$

where $\boldsymbol{\lambda} \equiv \lambda^{\mathbb{Z}^d}$,

$$U^X \equiv \sum_{\substack{A \in \mathfrak{F} \\ A \cap X \neq \varnothing}} J_A,$$

$$Z_X(\xi_{X^C}) \equiv \int_{\mathbf{M}} \exp[-U^X(\mathbf{y}_X \bullet \xi_{X^C})] \, \boldsymbol{\lambda}(d\mathbf{y})$$

Finally, we say that $\mu \in \mathfrak{M}_1(\mathbf{M})$ is a *Gibbs state for* $\mathfrak{E}(\mathscr{J})$ and write $\mu \in \mathfrak{G}(\mathscr{J})$ if

$$\langle \mathbb{E}^X \varphi, \mu \rangle = \langle \varphi, \mu \rangle \qquad \text{for all} \quad X \in \mathfrak{F} \text{ and } \varphi \in C_0(\mathbf{M})$$

Glauber Dynamics. Given a potential $\mathscr{J}$, we will say that the $\{P_t : t>0\}$ is a *Glauber dynamics for* $\mathscr{J}$ if $\{P_t : t>0\}$ is a Markov semigroup on $C(\mathbf{M})$ with the property that

$$(P_t\varphi, \psi)_{L^2(\mu)} = (P_t\psi, \varphi)_{L^2(\mu)} \qquad \text{for all} \quad \varphi, \psi \in C(\mathbf{M}) \tag{0.2}$$

There are various ways in which Glauber dynamics can be constructed. However, the type of conclusions reached in this article do not depend heavily on the particular choice. In fact, standard comparison arguments allow one to transfer these results proved for one choice of dynamics to other choices. Thus, we will restrict our attention to the dynamics which is determined by the operator $\mathscr{L}$ on $C_0(\mathbf{M})$ given by

$$\mathscr{L}\varphi = \sum_{\mathbf{k} \in \mathbb{Z}^d} (\mathbb{E}^{\{\mathbf{k}\}}\varphi - \varphi) \tag{0.3}$$

That is, $\mathscr{L}$ *determines* $\{P_t : t>0\}$ in the sense that there is precisely one Markov semigroup $\{P_t : t>0\}$ on $C(\mathbf{M})$ with the property that

$$P_t\varphi - \varphi = \int_0^t P_s \circ \mathscr{L}\varphi \, ds, \qquad t \in (0, \infty) \text{ and } \varphi \in C_0(\mathbf{M})$$

The Dirichlet Form. A key role in our analysis will be played by the *Dirichlet form* associated with our Glauber dynamics. Actually, one cannot describe a Dirichlet form for a Glauber dynamics without also specifying a Gibbs state. For this reason, we begin by defining the *local Dirichlet form* of our Glauber dynamics and will then get the actual Dirichlet form corresponding to a particular Gibbs state by integration with respect to that Gibbs state. Thus, set

$$\mathfrak{d}(\varphi, \psi)(\xi) = \frac{1}{2} \sum_{\mathbf{k} \in \mathbb{Z}^d} \int_{\mathbf{M}} (\varphi(\eta) - \varphi(\xi))(\psi(\eta) - \psi(\xi)) \, E^{\{\mathbf{k}\}}(d\eta \mid \xi)$$
$$\xi \in \mathbf{M} \qquad \text{and} \qquad \varphi, \psi \in C_0(\mathbf{M}) \tag{0.4}$$

It is then an easy matter to check that, for $\mu \in \mathfrak{G}(\mathscr{J})$, the Dirichlet form $\mathscr{E}_\mu$ associated with $\{P_t : t>0\}$ on $L^2(\mu)$ satisfies

$$\mathscr{E}_\mu(\varphi, \psi) \equiv -(\varphi, \mathscr{L}\psi)_\mu = \langle \mathfrak{d}(\varphi, \psi) \rangle_\mu \tag{0.5}$$

Remark. Everything that we do here could be done equally well in the setting of a continuous spin space when the Glauber dynamics is taken to be a diffusion. In fact, for technical reasons, the continuous setting is a little easier even though the underlying ideas are basically identical.

The purpose of this article is to clarify the connections between a potential $\mathscr{J}$, the class of Gibbs states $\mathfrak{G}(\mathscr{J})$, and Glauber dynamics associated with $\mathscr{J}$.

1. LOCAL SOBOLEV INEQUALITIES AND UNIFORM ERGODICITY

Given $\mu \in \mathfrak{G}(\mathscr{J})$ and $\mathbf{B} = \{\beta_{\mathbf{k}} : \mathbf{k} \in \mathbb{Z}^d\} \subseteq [0, \infty)$, we say that μ satisfies a *local logarithmic Sobolev inequality with coefficients* $\mathbf{B}$ if

$$\int_{\mathbf{M}} \varphi^2 \log \frac{\varphi^2}{\|\varphi\|_{2,\mu}^2} \, d\mu \leqslant \tfrac{1}{2} \sum_{\mathbf{k} \in \mathbb{Z}^d} \beta_{\mathbf{k}} \, \|\mathbb{E}^{\{\mathbf{k}\}}\varphi - \varphi\|_{2,\mu}^2, \qquad \varphi \in C_0(\mathbf{M}) \quad (1.1)$$

The terminology *local* is used to distinguish such an inequality from the situation when the set $\mathbf{B}$ is bounded, in which case one says that μ satisfies a *logarithmic Sobolev inequality*.

The purpose of this section is to see what can be said about the ergodic properties of associated Glauber dynamics when a local logarithmic Sobolev inequality holds and the coefficients have moderate growth (cf. Corollary 1.5 below).[3]

There are two ingredients in our program. The first of these is a general fact about Glauber dynamics corresponding to finite-range potentials and is a quantitative statement of the fact that, with very high probability, *disturbances propagate at a finite rate* in such dynamics. The second is an extension of the Gross's integration lemma.

Lemma 2.1. Let $R \in \mathbb{N}$ denote the range of interaction and set

$$\Lambda_n = \{\mathbf{k} \in \mathbb{Z}^d : |\mathbf{k}| \leqslant nR\} \qquad \text{for} \quad n \in \mathbb{N}$$

Then, there exists $A \in (0, \infty)$, depending only on R, d, and $\text{card}(M) + \|U^{\{\mathbf{0}\}}\|_{\mathbf{u}}$, such that, for each $m \in \mathbb{N}$,

$$\sum_{|\mathbf{k}| \geqslant (m+n)R} \|\partial_{\mathbf{k}} \circ P_t \varphi\|_{\mathbf{u}} \leqslant \mathbf{e}_n(At) \sum_{|\mathbf{k}| \leqslant mR} \|\partial_{\mathbf{k}} \varphi\|_{\mathbf{u}}, \qquad \varphi \in C_{\Lambda_m}(\mathbf{M}) \quad (1.2)$$

where

$$\mathbf{e}_n(s) \equiv e^s - \sum_{m=0}^{n-1} \frac{s^m}{m!} \leqslant e^s \left(\frac{se}{n}\right)^n \quad (1.3)$$

[3] Local logarithmic Sobolev inequalities have been considered previously in refs. 17 and 8. However, the goal in these articles was somewhat different from the one here.

In particular, A can be chosen so that, in addition, for each $q \in [2, \infty)$:

$$\|P_t \varphi\|_{\mathbf{u}} \leqslant \exp\left[\frac{A(m^d + n^d)}{q}\right] \left(2\mathbf{e}_n(At) \sum_{\mathbf{k} \in \Lambda_m} \|\partial_{\mathbf{k}} \varphi\|_{\mathbf{u}} + \|P_t \varphi\|_{q,\mu}\right) \quad (1.4)$$

Proof. The inequality in (1.2) is proved in Lemma 1.8 of ref. 15. To prove (1.4), for each $n \in \mathbb{Z}^+$, define

$$\mathscr{L}^{(n)} \varphi = \sum_{\mathbf{k} \in \Lambda_{n-1}} (\mathbb{E}^{\{\mathbf{k}\}} \varphi - \varphi) \qquad \text{for} \quad \varphi \in C(\mathbf{M})$$

It is then an elementary matter to show that $\mathscr{L}^{(n)}$ determines a Markov semigroup $\{P_t^{(n)} : t > 0\}$ on $C(\mathbf{M})$ which leaves $C_{\Lambda_n}(\mathbf{M})$ invariant. Moreover, using (1.2) and the reasoning in Lemma 1.8 of ref. 15, one can show that, with the same choice of $A \in (0, \infty)$ and all $m \in \mathbb{N}$ and $n \in \mathbb{Z}^+$,

$$\|P_t \varphi - P_t^{(m+n)} \varphi\|_{\mathbf{u}} \leqslant \mathbf{e}_n(At) \sum_{\mathbf{k} \in \Lambda_m} \|\partial_{\mathbf{k}} \varphi\|_{\mathbf{u}}, \qquad \varphi \in C_{\Lambda_m}(\mathbf{M}) \qquad (1.5)$$

In order to pass from (1.5) to (1.4), we need to know that there exists a $B \in (0, \infty)$ (with the required dependence) such that, for all $n \in \mathbb{Z}^+$ and $\varphi \in C_{\Lambda_{n-1}}(\mathbf{M})$:

$$\|\varphi\|_{\mathbf{u}} \leqslant \exp(Bn^d) \, \|\varphi\|_{1,\mu} \qquad (1.6)$$

Indeed, when M is finite, one has that

$$\|\varphi\|_{\mathbf{u}} \leqslant (\operatorname{card}(M))^{|\Lambda|} \exp[\operatorname{osc}(U^{\Lambda})] \, \|\varphi\|_{1,\mu}$$

for any $\Lambda \in \mathfrak{F}$ and $\varphi \in C_{\Lambda}(\mathbf{M})$.[4]

Given (1.5) and (1.6), one has, for any $q \in [2, \infty)$,

$$\begin{aligned}
\|P_t \varphi\|_{\mathbf{u}} &\leqslant \mathbf{e}_n(At) \sum_{\mathbf{k} \in \Lambda_m} \|\partial_{\mathbf{k}} \varphi\|_{\mathbf{u}} + \|P_t^{(m+n)} \varphi\|_{\mathbf{u}} \\
&\leqslant \mathbf{e}_n(At) \sum_{\mathbf{k} \in \Lambda_m} \|\partial_{\mathbf{k}} \varphi\|_{\mathbf{u}} + \exp\left[\frac{B(m+n)^d}{q}\right] \|P_t^{(m+n)} \varphi\|_{q,\mu} \\
&\leqslant \exp\left[\frac{B(m+n)^d}{q}\right] \left(2\mathbf{e}_n(At) \sum_{\mathbf{k} \in \Lambda_m} \|\partial_{\mathbf{k}} \varphi\|_{\mathbf{u}} + \|P_t \varphi\|_{q,\mu}\right)
\end{aligned}$$

where, in the passage to the second line, we have used (1.6) and elementary interpolation. Hence, (1.4) follows after one makes a minor adjustment. ∎

[4] This is the one place in which the continuous-spin setting is slightly more difficult. Namely, in that setting, (1.6) has to be replaced by an estimate in which the semigroup has been allowed to act for a little while. See Lemma 2.3 in ref. 15 for more details.

In order to demonstrate how we plan to use the contents of Lemma 1.1, we begin by considering the case when μ satisfies a logarithmic Sobolev inequality. That is, suppose that [cf. (0.5)]

$$\int_{\mathbf{M}} \varphi^2 \log \frac{\varphi^2}{\|\varphi\|_{2,\mu}^2} \leqslant \beta \mathscr{E}_\mu(\varphi, \varphi), \qquad \varphi \in C_0(\mathbf{M}) \tag{1.7}$$

for some $\beta \in (0, \infty)$. By Gross' integration lemma (cf. Theorem 6.1.4 in ref. 2), (1.7) implies that P_t maps $L^2(\mu)$ contractively into $L^{q(t)}(\mu)$, where $q(t) = 1 + \exp[2t/\beta]$ and (cf. Corollary 6.1.17, ref. 2) that one has the *spectral gap* estimate

$$m(\mu) \equiv \inf\{\mathscr{E}_\mu(\varphi, \varphi) : \|\varphi\|_{2,\mu} = 1 \text{ and } \langle \varphi \rangle_\mu = 0\rangle \geqslant 2/\beta \tag{1.8}$$

Note that, by elementary spectral theory, (1.8) is equivalent to

$$\|P_t \varphi - \langle \varphi \rangle_\mu \|_{2,\mu} \leqslant e^{-m(\mu) t} \| \varphi - \langle \varphi \rangle_\mu \|_{2,\mu} \tag{1.9}$$

for all $t \in (0, \infty)$ and $\varphi \in L^2(\mu)$. In order to replace these with statements about uniform convergence, let $\theta \in (0, 1)$ be given and taken $n = [t^2] + 1$ and $q = 1 + e^{2(1-\theta)t/\beta}$ in (1.4), and conclude that

$$\|P_t \varphi\|_{\mathbf{u}} \leqslant \exp\left[\frac{A(m^d + t^{2d})}{1 + e^{2(1-\theta)t/\beta}}\right] \left(2 \exp(At - t^2 \log At) \sum_{\mathbf{k} \in \Lambda_m} \|\partial_{\mathbf{k}} \varphi\|_{\mathbf{u}} + \|P_{\theta t} \varphi\|_{2,\mu}\right)$$

Thus, after replacing φ by $\varphi - \langle \varphi \rangle_\mu$ and using (1.9), we arrive at the conclusion that, for each $m \in \mathbb{N}$, there exists a $t(\theta, m) \in [1, \infty)$ such that

$$\|P_t \varphi - \langle \varphi \rangle_\mu \|_{\mathbf{u}} \leqslant 6 \exp[-\theta m(\mu) t] \sum_{\mathbf{k} \in \Lambda_m} \|\partial_{\mathbf{k}} \varphi\|_{\mathbf{u}}, \qquad \varphi \in \mathbf{C}_{\Lambda_m}(\mathbf{M}) \tag{1.10}$$

whenever $t \geqslant t(\theta, m)$. In particular, together with (1.8), this tells us the following.

Theorem 1.2. If $\mu \in \mathfrak{G}(\mathscr{J})$ satisfies a logarithmic Sobolev inequality, then $\mathfrak{G}(\mathscr{J}) = \{\mu\}$ and the associated Glauber dynamics acting on a local function φ converges uniformly to $\langle \varphi \rangle_\mu$ with exponential rate $m(\mu) > 0$.

In the remainder of this section we will show that uniform convergence of Glauber dynamics can sometimes be proved on the basis of a local logarithmic Sobolev inequality. However, in order to do so, we will require the following variant of Gross' integration lemma.

Lemma 1.3. Assume that (1.1) holds, and let $0 \leqslant T < t$ be given. Next, suppose that $q\colon [T, t] \to [2, \infty)$ is a differentiable, increasing function, and define

$$\Lambda(s) = \{\mathbf{k} \in \mathbb{Z}^d : \dot{q}(s)\,\beta_{\mathbf{k}} \geqslant 4(q(s) - 1)\} \qquad \text{for} \quad s \in [T, t]$$

Then

$$\|P_t\varphi\|^2_{q(t)} \leqslant \|P_T\varphi\|^2_{q(T)} + \int_T^t \dot{q}(s) \sum_{\mathbf{k} \in \Lambda(s)} \beta_{\mathbf{k}}\, \|\partial_{\mathbf{k}} P_s\varphi\|^2_{q(s)}\, ds, \qquad \varphi \in C_0(\mathbf{M}) \quad (1.11)$$

Proof. Let φ be a uniformly positive element of $C_0(\mathbf{M})$, and set $\varphi_s = P_s\varphi$. Then [cf. Theorem 6.1.14 in ref. 2 and (0.4)]

$$\begin{aligned}
&\frac{d}{ds}\|\varphi_s\|_{q(s),\mu} \\
&\quad \leqslant q(s)^{-2}\,\|\varphi_s\|^{1-q(s)}_{q(s),\mu} \left\{\dot{q}(s) \int \varphi_s^{q(s)} \log\left(\frac{\varphi_s}{\|\varphi_s\|_{q(s),\mu}}\right)^{q(s)} d\mu \right. \\
&\qquad \left. - 4(q(s)-1)\,\mathscr{E}_\mu(\varphi_s^{q(s)/2}, \varphi_s^{q(s)/2})\right\} \\
&\quad \leqslant \frac{\dot{q}(s)}{2q(s)^2}\,\|\varphi_t\|^{1-q(s)}_{q(s),\mu} \sum_{\mathbf{k} \in \Lambda(s,\mathbf{B})} \beta_{\mathbf{k}}\, \|\mathbb{E}^{\{\mathbf{k}\}}\varphi_s^{q(s)/2} - \varphi_s^{q(s)/2}\|^2_{2,\mu}
\end{aligned}$$

At the same time, for any $q \in [2, \infty)$ and positive $\varphi \in C(\mathbf{M})$

$$\begin{aligned}
&\|\mathbb{E}^{\{\mathbf{k}\}}\varphi^{q/2} - \varphi^{q/2}\|^2_{2,\mu} \\
&\quad = \iint (\varphi(\mathbf{y})^{q/2} - \varphi(\mathbf{x})^{q/2})^2\, \mathbb{E}^{\{\mathbf{k}\}}(d\mathbf{y} \mid \mathbf{x})\, \mu(d\mathbf{x}) \\
&\quad = 2\int \left\{\int_{\varphi(\mathbf{y}) < \varphi(\mathbf{x})} (\varphi(\mathbf{y})^{q/2} - \varphi(\mathbf{x})^{q/2})^2\, \mathbb{E}^{\{\mathbf{k}\}}(d\mathbf{y} \mid \mathbf{x})\right\} \mu(d\mathbf{x}) \\
&\quad \leqslant \frac{q^2}{2} \int \varphi(\mathbf{x})^{q-2} \left\{\int_{\varphi(\mathbf{y}) < \varphi(\mathbf{x})} (\varphi(\mathbf{y}) - \varphi(\mathbf{x}))^2\, \mathbb{E}^{\{\mathbf{k}\}}(d\mathbf{y} \mid \mathbf{x})\right\} \mu(d\mathbf{x}) \\
&\quad \leqslant \frac{q^2}{2}\,\|\varphi\|^{q-2}_{q,\mu} \left(\int \left\{\int_{\varphi(\mathbf{y}) < \varphi(\mathbf{x})} (\varphi(\mathbf{y}) - \varphi(\mathbf{x}))^2\, \mathbb{E}^{\{\mathbf{k}\}}(d\mathbf{y} \mid \mathbf{x})\right\}^{q/2} \mu(d\mathbf{x})\right)^{2/q} \\
&\quad \leqslant \frac{q^2}{2^{1+2/q}}\,\|\varphi\|^{q-2}_{q,\mu} \left\{\iint |\varphi(\mathbf{y}) - \varphi(\mathbf{x})|^q\, \mathbb{E}^{\{\mathbf{k}\}}(d\mathbf{y} \mid \mathbf{x})\, \mu(d\mathbf{x})\right\}^{2/q} \\
&\quad \leqslant 2^{1-2/q} q^2\,\|\varphi\|^{q-2}_{q,\mu}\, \|\partial_{\mathbf{k}}\varphi\|^2_{q,\mu}
\end{aligned}$$

Hence, after plugging this into the preceding and integrating, we arrive at (1.11). ∎

Theorem 1.4. Let $A \in (0, \infty)$ be chosen so that (1.2) and (1.4) hold, and assume that (1.1) holds with some $\mathbf{B}$ satisfying

$$\sup\left\{\beta_{\mathbf{k}} \exp\left(-\frac{|\mathbf{k}|}{2R}\right) : \mathbf{k} \in \mathbb{Z}^d\right\} < \infty$$

Next, let ρ: $[0, \infty) \to (0, \infty)$ be a continuous, nonincreasing function which tends to 0 at ∞ and satisfies

$$\sup_{|\mathbf{k}| \geqslant n} \rho(|\mathbf{k}|)\, \beta_{\mathbf{k}} < 4 \qquad \text{for some} \quad n \in \mathbb{N}$$

Finally, choose $\kappa > \kappa_0 \equiv (e^2 RA) \vee 1$, and set

$$T_0 = \min\{s \geqslant 0 : \rho(\kappa s)\, \beta_{\mathbf{k}} < 4 \text{ for all } |\mathbf{k}| \leqslant n\}$$

Then there exists an $M \in [1, \infty)$ such that, for any $m \in \mathbb{N}$, $T \geqslant T_m \equiv T_0 \vee mR/(\kappa - \kappa_0)$, and $t > T$:

$$\|P_t \varphi\|_u \leqslant \exp\left(\frac{M(m^d + t^d)}{q}\right)\left[\|P_T \varphi\|_{2,\mu} + M e^{-At} \sum_{\mathbf{k} \in \Lambda_m} \|\partial_k \varphi\|_u\right]$$
$$\text{for all} \quad \varphi \in C_{\Lambda_m}(\mathbf{M}) \tag{1.12}$$

where

$$q \equiv 1 + \exp\left\{\int_T^t \rho(\kappa\tau)\, d\tau\right\}$$

Proof. Choose n to be the smallest integer which dominates $e^2 At$. Then, by (1.4), we find that

$$\|P_t \varphi\|_{\mathbf{u}} \leqslant e^{-At} \sum_{\mathbf{k} \in \Lambda_m} \|\partial_{\mathbf{k}} \varphi\|_{\mathbf{u}} + \exp\left(\frac{A(m + 1 + e^2 At)^d}{q}\right) \|P_t \varphi\|_{q,\mu} \tag{1.13}$$

At the same time, by (1.11) with $q(s) = 1 + \exp\{\int_T^s \rho(\kappa\tau)\, d\tau\}$,

$$\|P_t \varphi\|_{q,\mu} \leqslant \|P_T \varphi\|_{2,\mu} + \left\{\int_T^t \dot{q}(s) \sum_{\mathbf{k} \in \Lambda(s)} \beta_{\mathbf{k}} \|\partial_{\mathbf{k}} P_s \varphi\|_{\mathbf{u}}^2\, ds\right\}$$

But, for $s \geqslant T_m$, $\mathbf{k} \in \Lambda(s) \Rightarrow |\mathbf{k}| \geqslant mR + \kappa_0 s$, and so, by (1.2),

$$\sum_{\mathbf{k} \in \Lambda(s)} \beta_{\mathbf{k}} \|\partial_{\mathbf{k}} P_s \varphi\|_{\mathbf{u}}^2 \leqslant e^{-As}\Big(\sup_{|l| \geqslant mR + \kappa_0 s} \beta_l \|\partial_l P_s \varphi\|_{\mathbf{u}}\Big)\left(\sum_{\mathbf{k} \in \Lambda_m} \|\partial_{\mathbf{k}} \varphi\|_{\mathbf{u}}\right)$$

Finally, after another application of (1.2), one finds that, for all $l \geqslant mR + \kappa_0 s$,

$$\|\partial_l P_s \varphi\|_{\mathbf{u}} \leqslant \exp\left(\frac{m}{2} - \frac{|l|}{2R}\right)$$

Hence, after combining this with (1.13), it is an easy matter to find an M for which (1.12) holds. ∎

Corollary 1.5. Let everything be as in the statement of Theorem 1.4, and assume that, for some $\kappa > \kappa_0$,

$$\lim_{t \to \infty} \left\{ \int_1^t \rho(\tau)\, d\tau - d\kappa \log t \right\} = \infty \tag{1.14}$$

Then $\mathfrak{G}(\mathscr{J}) = \{\mu\}$, and

$$\lim_{t \to \infty} \|P_t \varphi - \langle \varphi \rangle_\mu\|_{\mathbf{u}} = 0 \qquad \text{for all} \quad \varphi \in C(\mathbf{M})$$

In fact, if $t_m \in [T_m, \infty)$ is determined by $\int_{T_m}^{t_m} \rho(\kappa\tau)\, d\tau = d \log t_m$ and $[t_0, \infty) \mapsto T_\kappa(t) \in [1, \infty)$ is defined by

$$\int_{T_k(t)}^t \rho(\kappa\tau)\, d\tau = d \log t \tag{1.15}$$

then, for $m \in \mathbb{N}$ and $t \in [t_m, \infty)$,

$$\|P_t \varphi - \langle \varphi \rangle_\mu\|_{\mathbf{u}} \leqslant K(t, m) \left(\|P_{T_\kappa(t)} \varphi - \langle \varphi \rangle_\mu\|_{2,\mu} + M e^{-At} \sum_{\mathbf{k} \in \Lambda_m} \|\partial_{\mathbf{k}} \varphi\|_{\mathbf{u}} \right)$$

$$\text{for} \quad \varphi \in C_{\Lambda_m}(\mathbf{M}),$$

$$\text{where} \quad K(t, m) \equiv \exp\left[M \left(1 + \frac{m^d}{t^d} \right) \right] \tag{1.16}$$

Proof. Given $m \in \mathbb{N}$, choose T_m as in Theorem 1.4, let $T > T_m$, and take

$$q(t) = 1 + \exp\left[\int_T^t \rho(\kappa)\, d\tau \right], \qquad t > T$$

and observe that, from (1.14) combined with (1.12), one gets first (1.16) and then

$$\overline{\lim_{t \to \infty}} \|P_t \varphi\|_{\mathbf{u}} \leqslant \|P_T \varphi\|_{2,\mu}, \qquad \varphi \in C_{\Lambda_m}(\mathbf{M}) \tag{1.17}$$

Hence, if E_0^μ denotes the orthogonal projection onto the subspace of $L^2(\mu)$ which is invariant under extension to $L^2(\mu)$ of the semigroup $\{P_t : t > 0\}$, then, after letting $T \nearrow \infty$ and noting that $P_T\varphi$ tends to $E_0^\mu\varphi$ in $L^2(\mu)$, we obtain

$$\overline{\lim_{t\to\infty}} \|P_t\varphi\|_{\mathbf{u}} \leqslant \|E_0^\mu\varphi\|_{2,\mu}, \qquad \varphi \in C_0(\mathbf{M})$$

as an easy application of the Spectral Theorem for self-adjoint semigroups of contractions. But this means that, for any pair of φ and ψ from $C_0(\mathbf{M})$,

$$(E_0^\mu\varphi, \psi)_{L^2(\mu)} \leqslant \|E_0^\mu\varphi\|_{2,\mu} \|\psi\|_{1,\mu}$$

from which it is an easy step first to

$$\|E_0^\mu\varphi\|_{\infty,\mu} \leqslant \|E_0^\mu\varphi\|_{2,\mu}$$

and then to the conclusion that $E_0^\mu\varphi = \langle\varphi\rangle_\mu$ for all $\varphi \in L^2(\mu)$. In particular, we now know that, for each $\varphi \in C_0(\mathbf{M})$, $P_t\varphi \to \langle\varphi\rangle_\mu$ in $L^2(\mu)$, which, by (1.15) leads to asserted uniform convergence first for $\varphi \in C_0(\mathbf{M})$ and thence for all $\varphi \in C(\mathbf{M})$. ∎

Corollary 1.6. Again let everything be as in Theorem 1.4, with a choice of $A \in (m(\mu), \infty)$ [cf. (1.8)]. If

$$\overline{\lim_{t\to\infty}}\, t\rho(t) \geqslant \frac{\kappa d}{1-\theta} \qquad \text{for some} \quad \theta \in (0, 1)$$

then, for each $\varphi \in C_0(\mathbf{M})$, there exists a $T(\varphi) \in (0, \infty)$ such that

$$\|P_t\varphi - \langle\varphi\rangle_\mu\|_{\mathbf{u}} \leqslant 2\, \|P_{t^\theta}\varphi - \langle\varphi\rangle_\mu\|_{2,\mu} \qquad \text{for all} \quad t \geqslant T(\varphi)$$

and so, if $m(\mu) > 0$, then

$$\overline{\lim_{t\to\infty}}\, t^{-\theta} \log \|P_t\varphi - \langle\varphi\rangle_\mu\|_{\mathbf{u}} \leqslant -m(\mu)$$

Moreover, if

$$\overline{\lim_{t\to\infty}} \frac{t\rho(t)}{\log t} \geqslant \varepsilon\kappa \qquad \text{for some} \quad \varepsilon > 0$$

and $\alpha \equiv \exp(-2d/\varepsilon)$, then, for each $\varphi \in C_0(\mathbf{M})$, there exists a $T(\varphi) \in (0, \infty)$ such that

$$\|P_t\varphi - \langle\varphi\rangle_\mu\|_{\mathbf{u}} \leqslant 2\, \|P_{\alpha t}\varphi - \langle\varphi\rangle_\mu\|_{2,\mu} \qquad \text{for all} \quad t \geqslant T(\varphi)$$

and so, if $m(\mu) > 0$, then

$$\overline{\lim_{t \to \infty}}\ t^{-1} \log \|P_t \varphi - \langle \varphi \rangle_\mu\|_u \leqslant -\alpha m(u)$$

Remark 1.7. In connection with the results in Corollary 1.6, it should be kept in mind that, at least for attractive systems, Holley[6] has shown that

$$\lim_{t \to \infty} t^{\beta} \|P_t \varphi - \langle \varphi \rangle_\mu\|_u = 0 \qquad \text{for some} \quad \beta > d \quad \text{and all } \varphi \in C_0(\mathbf{M})$$

already implies that uniform convergence is taking place at an exponential rate.

2. SUMMARY

In this paper we have demonstrated that if any Gibbs measure satisfies a logarithmic Sobolev inequality, then that Gibbs state is the only one and, for local functions, the corresponding (Glauber) stochastic dynamics converges to it in the uniform norm at an exponentially rapid rate. This provides a uniqueness and ergodicity criterion which is based directly on the *infinite-volume* considerations, one which is therefore *a priori* free of anything having to do with finite-volume systems and their boundary conditions. (For some recent results involving finite-volume considerations, see refs. 14–16 and 8–13). Besides its esthetic value, this criterion may prove useful when it comes to understanding situations where the Dobrushin–Shlosman[4,5] complete analyticity fails, but one still has some kind of good behavior on the large scale.

Second, we have shown that a bona fide logarithmic Sobolev inequality can be replaced by a local version in which the coefficients have moderate growth (as a function of the distance from the origin). This observation generalizes a result in ref. 7, where sublinear growth of a finite-volume logarithmic Sobolev coefficient was considered. With the results proved here, one sees that linear growth is permissible as long as the slope is sufficiently small. In addition, these considerations provide an elegant framework for the description of the stochastic dynamics associated with spin systems having random interactions (cf. ref. 18, for example). Finally, we believe that they may also apply to other nonferromagnetic spin systems in which the strong analyticity conditions fail (e.g., the system discussed in ref. 3). For closely related results, see ref. 1; and to see how much better one can do in the ferromagnetic situation, see Part I of ref. 10.

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